

# Quantum strategies of quantum measurements

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In classical Monty Hall problem, one player can always win with probability  $2/3$ . We generalize the problem to the quantum domain and show that a fair two-party zero-sum game can be carried out if the other player is permitted to adopt quantum measurement strategy.

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Recently, quantum game theory [1–5] has drawn much attentions, which is the combination of applied mathematics and quantum information theory. It has been shown that quantum strategies can be more successful than classical ones, based on the principles of quantum mechanics, such as quantum entanglement [1,2] and quantum superposition [3,4]. Here we show that quantum measurement, which has been successful in lots of quantum processes, such as quantum programming [6], the purification of entanglements [7], probabilistic teleportation [8] and cloning [9], can also contribute to quantum strategies.

A novel quantum game has been introduced by Goldenberg *et al.* [4], where a particle and two boxes are involved. Here, we consider the Monty Hall problem [10], a two-party zero-sum game involving a particle and more boxes.

In the classical case, Alice may put the particle in one of some (such as three) boxes and Bob picks one box. If he finds the particle in this box, he wins (suppose one coin), otherwise, he loses one coin. Obviously, Bob will not agree to this proposal, for he has definite  $2/3$  chance to lose. He may argue that after he chooses one box, Alice should reveal *an empty one from the other two* boxes and then he is provided a chance to choose between sticking with the original choice and switching to the third box. Counterintuitively, this puts them in a dilemma situation. Bob can win with probability  $2/3$  by choosing to switch [10,11]. The key point is that when Bob selects a box, he has expected  $1/3$  chance to win, which will not change anymore. Under the condition that Alice reveals an empty box, Bob may have  $2/3$  chance to find the particle in the third box.

Inspired by the previous nice works [1–4], we may ask if Alice can change the situation and play a fair game with Bob provided with quantum strategies.

A quantum version of Monty Hall problem may be as follows: there are one quantum particle and three boxes 0, 1 and 2 (the quantum delineation is a particle with three eigenstates  $|0\rangle$ ,  $|1\rangle$  and  $|2\rangle$ ). Alice puts the particle into the boxes (maybe in a superposition state). Bob picks one box. In order to determine the location of the particle, Alice may measure the particles herself before she reveals an empty box to Bob. At last Bob makes his decision on sticking or switching.

Obviously, in the classical problem, Alice is forbidden to pick the particle to other boxes after he places it (otherwise, it becomes an ordinary gambling with two boxes). It's a natural generalization that in the quantum case Alice cannot evolve the particle unitarily even if she has an auxiliary particle, that is (i) single bit rotations (ii) controlled-NOT operations [12] on the particle are prohibited. While both in classical and quantum problem, measurement is permitted.

We first consider the quantum superposition and entanglement strategies. Alice may prepare the particle in a superposition state, such as

$$|\psi\rangle_p = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle). \quad (1)$$

After his selection, Bob has  $1/3$  chance to win. According to rule (i), Alice cannot change the probability distribution (quantum measurement strategy will be discussed below). After Alice points out an empty box (von Neumann measurement under the basis  $\{|0\rangle, |1\rangle, |2\rangle\}$  is needed), Bob still has  $2/3$  chance to win by switching. Alice can also entangle the particle with an auxiliary particle, i.e., the particles are in an entangled state, such as

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$$|\Psi\rangle_{p,aux} = \frac{1}{\sqrt{3}} (|0\rangle|0\rangle_{aux} + |1\rangle|1\rangle_{aux} + |2\rangle|2\rangle_{aux}). \quad (2)$$

According to the rules (i) and (ii), what Alice can do is only local operations on the auxiliary particle. Obviously, these operations cannot affect the local probability distribution of the playing particle, which means the dilemma situation remains unchanged.

Now, we suppose that Alice is allowed to adopt the quantum measurement strategies. For simplicity, we first assume that Alice put the particle in one of the Boxes initially. The box Bob chooses is assigned  $|0\rangle$ , and Alice reveals no particle in  $|2\rangle$ . At this moment, the state of the particle may be described by

$$\rho_p = \frac{1}{3} |0\rangle\langle 0| + \frac{2}{3} |1\rangle\langle 1|. \quad (3)$$

One property of quantum measurement is that there exists collective measurement on the Boxes. Because Alice knows exactly which box the particle is in, his strategy may depend on the choice of Bob. For the cases the particle in  $|0\rangle$  and in  $|1\rangle$ , suppose that the orthogonal bases of the von Neumann measurement Alice adopts are

$$\begin{aligned} |\phi\rangle_0 &= \cos \alpha_0 |0\rangle + \sin \alpha_0 |1\rangle, \\ |\phi\rangle_0^\perp &= -\sin \alpha_0 |0\rangle + \cos \alpha_0 |1\rangle, \end{aligned} \quad (4)$$

and

$$\begin{aligned} |\phi\rangle_1 &= \cos \alpha_1 |0\rangle + \sin \alpha_1 |1\rangle, \\ |\phi\rangle_1^\perp &= -\sin \alpha_1 |0\rangle + \cos \alpha_1 |1\rangle, \end{aligned} \quad (5)$$

respectively, where  $0 \leq \alpha_0, \alpha_1 \leq \frac{\pi}{2}$ . The density operator of the particle becomes

$$\rho'_p = \frac{1}{3} \left( \cos^2 \alpha_0 |\phi\rangle_0 \langle \phi| + \sin^2 \alpha_0 |\phi\rangle_0^\perp \langle \phi|^\perp \right) + \frac{2}{3} \left( \cos^2 \alpha_1 |\phi\rangle_1 \langle \phi| + \sin^2 \alpha_1 |\phi\rangle_1^\perp \langle \phi|^\perp \right). \quad (6)$$

Bob's two pure strategies, sticking on  $|0\rangle$ , and switching to  $|1\rangle$ , are denoted by  $N$  and  $V$ , respectively, satisfying

$$N\rho = \langle 0|\rho|0\rangle, \quad V\rho = \langle 1|\rho|1\rangle. \quad (7)$$

We first confine Bob's strategies in a classical mixture region, that is

$$S_B(\eta) = \eta N + (1 - \eta) V, \quad (8)$$

where  $0 \leq \eta \leq 1$ . The expected probability for Bob to win is

$$\begin{aligned} P_B = S_B(\eta) \rho'_p &= \frac{1}{3} \left[ \eta (\cos^4 \alpha_0 + \sin^4 \alpha_0) + 2(1 - \eta) \sin^2 \alpha_0 \cos^2 \alpha_0 \right] \\ &+ \frac{2}{3} \left[ (1 - \eta) (\cos^4 \alpha_1 + \sin^4 \alpha_1) + 2\eta \sin^2 \alpha_1 \cos^2 \alpha_1 \right]. \end{aligned} \quad (9)$$

The corresponding expected gain for Bob is

$$G_B = 2P_B - 1. \quad (10)$$

It is easy to find the Nash equilibrium [13] of the problem: Alice adopts a deterministic strategy

$$\begin{aligned} |\phi\rangle_0 &= |\phi\rangle_1 = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \\ |\phi\rangle_0^\perp &= |\phi\rangle_1^\perp = \frac{1}{\sqrt{2}} (-|0\rangle + |1\rangle), \end{aligned} \quad (11)$$

while Bob adopts a probabilistic strategy

$$S_B\left(\frac{1}{2}\right) = \frac{1}{2} (N + V). \quad (12)$$

The equilibrium expected winning probability and gain of Bob are

$$P_B^e = \frac{1}{2}, G_B^e = 0. \quad (13)$$

During the deducing of the Nash equilibrium, our discussion is limited in a simple case, i.e., initially, Alice puts the particle in one of the three boxes and Bob chooses between sticking and switching at the end. Next, we simply show that the Nash equilibrium is stable in their whole strategic spaces: Alice may put the particle in a superposition state, while Bob may choose the quantum superposition of sticking and switching  $R(\beta)$ , with

$$R(\beta)\rho = (\cos\beta \langle 0| + \sin\beta \langle 1|)\rho(\cos\beta |0\rangle + \sin\beta |1\rangle). \quad (14)$$

If Alice adopts the strategy of Equations (11), according to Eq. (6), the density operator of the particle becomes

$$\rho'_p = I. \quad (15)$$

No matter what strategy Bob adopts, pure or mixture of Eq. (14),  $G_B \leq G_B^e = 0$  is always satisfied. On the other hand, if Bob adopts the strategy of Eq. (12), then,

$$S_B\left(\frac{1}{2}\right)\rho = \frac{1}{2}(N+V)\rho = \frac{1}{2}\text{tr}\rho \equiv \frac{1}{2}, \quad (16)$$

which is independent of Alice's strategy (the density operator of the particle). Therefore, neither of them can improve his/her expected gain (0 in this case) by changing his/her strategy while the other player does not. That is, the pair of strategies of Eqs. (11) and (12) is a Nash equilibrium.

Obviously, there is no deterministic Nash equilibrium, the reason lies in that Alice's strategy has two variable while Bob's pure strategy has one.

The power of the quantum measurement strategy may be embodied more clearly in the generalized  $N$ -stage Monty Hall problem, which can be delineated as follows: Step 1, Alice puts a particle in one of  $N$  boxes. Bob picks one box. Step 2, Alice reveals an empty one from the other  $N-1$  boxes and gives Bob the option of switching to one of the other  $N-2$  boxes. Step 3, Alice reveals another empty boxes, etc. After  $N$  stages, the gambling finishes. The best classical strategy of Bob is: stick until the last choice, then switch. The optimal expected winning probability is  $\frac{N-1}{N}$ . Obviously, Bob will definitely win, provided  $N \rightarrow \infty$ . However, if Alice is allowed to adopt quantum measurement strategy, it is easy to verify that there still exists a Nash equilibrium: Alice waits until the last step to adopt the strategy of Eqs. (11), while Bob sticks until the last choice to adopt the strategy of Eq. (12). Their expected gains are both 0, which means Alice can still play a fair game with Bob by adopting quantum measurement strategy. The main reason is that in quantum measurement, there exists collective measurement, while in classical measurement, there's only orthogonal measurement.

In conclusion, we have generalized the Monty Hall problem to the quantum domain. It is shown that a fair two-party zero-sum game can be carried out if a player is permitted to adopt quantum measurement strategy, while in classical situation, the other player can always win with high probability.

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